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MOTION OF A PULSATING RIGID BODY IN AN OSCILLATING VISCOUS FLUID

V. L. Sennitskii

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The motion of a rigid sphere in a viscous fluid due to specified pulsations of the sphere and specified oscillations of the fluid away from the sphere is considered.

The experimental results of [1] demonstrated the existence of predominantly unidirectional motion of a compressible rigid body in an oscillating fluid. The essence of this phenomenon is as follows. A compressible rigid body placed in a fluid in a closed container moves in a specified direction as a result of prescribed oscillations and deformations of the container. (Depending on the character of oscillations and deformations of the container, the body moves in the positive or negative direction of the axis along which the container oscillates.) Predominantly unidirectional motion of a compressible rigid body in an oscillating fluid is similar to predominantly unidirectional motion of a gas bubble in an oscillating fluid [2–4] and can be explained similarly [1–3]. However, in contrast with the motion of a gas bubble, which has been studied theoretically [2, 4], there have not been corresponding theoretical studies of the motion of a compressible rigid body. In the present paper, we consider the motion of a rigid sphere in a viscous fluid under conditions similar to the experimental conditions of [1].

1. A compressible rigid sphere is placed in an unbounded viscous incompressible fluid. The sphere radius and the fluid velocity at infinity relative to the inertial Cartesian coordinate system X, Y, Z change periodically with time t in a prescribed manner with period T (the average fluid velocity at infinity is zero). The distribution of the material of the sphere is symmetric about its center (the center of inertia coincides with the center of the sphere). The fluid flow is independent of the initial conditions. The position of the sphere is characterized by the radius-vector $\boldsymbol{S}$ of its center. The problem is to find $\boldsymbol{S}$ as a function of t.

This formulation of the problem corresponds to the following: a closed container is filled with a fluid and contains a body (a compressible rigid sphere), the walls of the container are at very large distances from the body, some of the walls are deformable, the container performs specified translatory oscillations and deforms in a specified manner, the deformations of the container cause the body to pulsate in a prescribed manner (the body volume changes periodically with time), and on the body surface, the fluid pressure variations due to the oscillations of the container are small compared to the fluid pressure variations due to the deformations of the container.

We consider the fluid flow and the motion of the body relative to the Cartesian coordinate system $X_1 = X - S_X$, $X_2 = Y - S_Y$, $X_3 = Z - S_Z$ (S_X, S_Y, and S_Z are, respectively, the X, Y, and Z components of the vector $\boldsymbol{S}$).

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We assume that $\tau = t/T$, $A = A_0(1 + æa)$ is the radius of the sphere $\Big[(A_0$ $(A_0 > 0)$ is a constant, $æ$ $(æ < 1)$ is the maximum value of $|A - A_0|/A_0$, and $a = \text{Real} \sum_{m=1}^{\infty} a_m e^{2m\pi i\tau}$ (a_m are constants)$\Big]$, $\boldsymbol{U} = \hat{U}\boldsymbol{u} = \hat{U}u\boldsymbol{k}$ is the fluid velocity at infinity $\Big[\hat{U}$ is the maximum value of $|\boldsymbol{U}|$, $u = \text{Real} \sum_{m=1}^{\infty} u_m e^{2m\pi i\tau}$ (u_m are constants), and $\boldsymbol{k} = (0, 0, 1)\Big]$, $x_1 = X_1/A_0$, $x_2 = X_2/A_0$, and $x_3 = X_3/A_0$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\varepsilon = \hat{U}T/A_0$, ρ is the density of the fluid, m is the mass of the sphere, $\mu = 3m/(4\pi A_0^3\rho)$, (s) is the surface of the sphere [the equation of (s) is given by $r = 1 + æa$], $\boldsymbol{n}$ is the unit outward normal to (s), $\boldsymbol{V}$ is the fluid velocity, $\boldsymbol{v} = T\boldsymbol{V}/A_0$, P is the fluid pressure, $p = T^2P/(\rho A_0^2)$, $\boldsymbol{w} = (1/A_0)\, d\boldsymbol{S}/d\tau$, ν is the kinematic viscosity of the fluid, $\text{Re} = A_0^2/(\nu T)$ is the Reynolds number, $\mathcal{P}$ is the stress tensor for the fluid, $\wp = T^2\mathcal{P}/(\rho A_0^2)$, $\boldsymbol{F}$ is the force exerted by the fluid on the body, and $\boldsymbol{f} = T^2\boldsymbol{F}/(\rho A_0^4) = \iint_{(s)} \wp \cdot \boldsymbol{n}\, ds$.

The equation of motion for the center of inertia of the body, the Navier–Stokes and continuity equations, and the conditions that must be satisfied at (s) and for $r \to \infty$ are written as

$$\boldsymbol{f} - \frac{4\pi}{3}\mu\frac{d\boldsymbol{w}}{d\tau} = 0, \qquad \frac{\partial\boldsymbol{v}}{\partial\tau} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{1}{\text{Re}}\Delta\boldsymbol{v} - \frac{d\boldsymbol{w}}{d\tau}, \qquad \nabla\cdot\boldsymbol{v} = 0; \tag{1.1}$$

$$\boldsymbol{v} = æ\frac{da}{d\tau}\boldsymbol{n} \quad \text{on} \quad (s), \qquad \boldsymbol{v} \sim \varepsilon\boldsymbol{u} - \boldsymbol{w} \quad \text{as} \quad r \to \infty. \tag{1.2}$$

2. We consider problem (1.1), (1.2) for small (compared to unity) values of ε.

We assume that as $\varepsilon \to 0$,

$$\boldsymbol{v} \sim \boldsymbol{v}^{(0)} + \varepsilon\boldsymbol{v}^{(1)}, \quad p \sim p^{(0)} + \varepsilon p^{(1)}, \quad \boldsymbol{w} \sim \boldsymbol{w}^{(0)} + \varepsilon\boldsymbol{w}^{(1)}. \tag{2.1}$$

According to (1.1), (1.2), and (2.1), in the Mth ($M = 0$, 1) approximation, we have

$$\boldsymbol{f}^{(M)} - \frac{4\pi}{3}\mu\frac{d\boldsymbol{w}^{(M)}}{d\tau} = 0,$$

$$\frac{\partial\boldsymbol{v}^{(M)}}{\partial\tau} + (\boldsymbol{v}^{(0)}\cdot\nabla)\boldsymbol{v}^{(M)} + M(\boldsymbol{v}^{(1)}\cdot\nabla)\boldsymbol{v}^{(0)} + \nabla p^{(M)} - \frac{1}{\text{Re}}\Delta\boldsymbol{v}^{(M)} + \frac{d\boldsymbol{w}^{(M)}}{d\tau} = 0, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{v}^{(M)} = 0;$$

$$\boldsymbol{v}^{(M)} = (1 - M)æ\frac{da}{d\tau}\boldsymbol{n} \quad \text{for} \quad r = 1 + æa, \qquad \boldsymbol{v}^{(M)} \sim M\boldsymbol{u} - \boldsymbol{w}^{(M)} \quad \text{as} \quad r \to \infty, \tag{2.3}$$

where $\boldsymbol{f}^{(M)} = \iint_{(s)} \wp^{(M)}\cdot\boldsymbol{n}\, ds$ [$\wp^{(M)}$ is $\wp$ taken at $\boldsymbol{v} = \boldsymbol{v}^{(M)}$ and $p = p^{(M)}$].

Let $M = 0$. For $\varepsilon = 0$, the center of inertia of the sphere is at rest relative to the coordinate system X, Y, Z, and the fluid flow is symmetric about the origin of the coordinates x_1, x_2, x_3. Problem (2.2) and (2.3) has the solution

$$\boldsymbol{v}^{(0)} = (1 + æa)^2 æ\frac{da}{d\tau}\frac{\boldsymbol{r}}{r^3}; \tag{2.4}$$

$$p^{(0)} = \frac{(1 + æa)^2}{r}\Big\{æ\frac{d^2a}{d\tau^2} + \frac{2æ^2}{1 + æa}\Big(\frac{da}{d\tau}\Big)^2\Big[1 - \frac{(1 + æa)^3}{4r^3}\Big]\Big\} + c^{(0)};$$

$$\boldsymbol{w}^{(0)} = 0. \tag{2.5}$$

Here $\boldsymbol{r} = (x_1, x_2, x_3)$ and $c^{(0)}$ is a function of τ.

Let $M = 1$. We consider problem (2.2) and (2.3) for small (compared to unity) values of æ (values of ε are small in comparison to æ).

From (2.4), it follows that for æ $\to 0$ we have

$$\boldsymbol{v}^{(0)} \sim æ\boldsymbol{v}^{(0)}_{(1)}, \tag{2.6}$$

where $\boldsymbol{v}^{(0)}_{(1)} = \dfrac{da}{d\tau}\dfrac{\boldsymbol{r}}{r^3}$.

We assume that for æ $\to 0$,

$$\boldsymbol{v}^{(1)} \sim \boldsymbol{v}^{(1)}_{(0)} + æ\boldsymbol{v}^{(1)}_{(1)}, \quad p^{(1)} \sim p^{(1)}_{(0)} + æp^{(1)}_{(1)}, \quad \boldsymbol{w}^{(1)} \sim \boldsymbol{w}^{(1)}_{(0)} + æ\boldsymbol{w}^{(1)}_{(1)}. \tag{2.7}$$

According to (2.2), (2.3), (2.6), and (2.7), in the Nth ($N = 0,\, 1$) approximation, we have

$$\boldsymbol{f}^{(1)}_{(N)} - \frac{4\pi}{3}\,\mu\,\frac{d\boldsymbol{w}^{(1)}_{(N)}}{d\tau} = 0,$$

$$\frac{\partial \boldsymbol{v}^{(1)}_{(N)}}{\partial\tau} + \nabla p^{(1)}_{(N)} - \frac{1}{\mathrm{Re}}\,\Delta\boldsymbol{v}^{(1)}_{(N)} + \frac{d\boldsymbol{w}^{(1)}_{(N)}}{d\tau} = -N\Big[(\boldsymbol{v}^{(0)}_{(1)}\cdot\nabla)\boldsymbol{v}^{(1)}_{(0)} + (\boldsymbol{v}^{(1)}_{(0)}\cdot\nabla)\boldsymbol{v}^{(0)}_{(1)}\Big], \tag{2.8}$$

$$\nabla\cdot\boldsymbol{v}^{(1)}_{(N)} = 0;$$

$$\boldsymbol{v}^{(1)}_{(N)} = -N\,\frac{\partial\boldsymbol{v}^{(1)}_{(0)}}{\partial r}\,a \quad \text{for} \quad r = 1, \qquad \boldsymbol{v}^{(1)}_{(N)} \sim (1-N)\boldsymbol{u} - \boldsymbol{w}^{(1)}_{(N)} \quad \text{as} \quad r\to\infty, \tag{2.9}$$

where $\boldsymbol{f}^{(1)}_{(N)} = \displaystyle\iint_{(s)} \wp^{(1)}_{(N)}\cdot\boldsymbol{n}\,ds$ [$\wp^{(1)}_{(N)}$ is $\wp$ for $\boldsymbol{v} = \boldsymbol{v}^{(1)}_{(N)}$, $p = p^{(1)}_{(N)}$].

Let $N = 0$. Problem (2.8), (2.9) has the solution

$$v^{(1)}_{(0)r} = \frac{1}{r^2\sin\theta}\,\frac{\partial\psi_{(0)}}{\partial\theta}, \qquad v^{(1)}_{(0)\theta} = -\frac{1}{r\sin\theta}\,\frac{\partial\psi_{(0)}}{\partial r}, \qquad v^{(1)}_{(0)\varphi} = 0;$$

$$p^{(1)}_{(0)} = \Big\{\Big[-\frac{\partial^2}{\partial\tau\,\partial r} + \frac{1}{\mathrm{Re}}\Big(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\,\frac{\partial}{\partial r} + \frac{4}{r^3}\Big)\Big]\psi_{(0)} - \frac{dw_{(0)}}{d\tau}\,r\sin^2\theta\Big\}\frac{\cos\theta}{\sin^2\theta} + c_{(0)};$$

$$\boldsymbol{w}^{(1)}_{(0)} = w_{(0)}\boldsymbol{k}. \tag{2.10}$$

Here $v^{(1)}_{(0)r}$, $v^{(1)}_{(0)\theta}$, and $v^{(1)}_{(0)\varphi}$ are, respectively, the r, θ, and φ components of the vector $\boldsymbol{v}^{(1)}_{(0)}$ [θ is the angle between the vectors $(0,0,1)$ and (x_1,x_2,x_3), and φ is the angle between the vectors $(1,0,0)$ and $(x_1,x_2,0)$],

$$\psi_{(0)} = \Big\{\frac{1}{2}\,(u - w_{(0)})r^2 + \frac{1}{2}\,\mathrm{Real}\sum_{m=1}^{\infty}\frac{w_{(0)m} - u_m}{q_m^2} \times \Big[\frac{q_m^2 + 3q_m + 3}{r} - 3\Big(q_m + \frac{1}{r}\Big)\mathrm{e}^{q_m(1-r)}\Big]\mathrm{e}^{2m\pi i\tau}\Big\}\sin^2\theta,$$

$c_{(0)}$ is a function of τ, and $w_{(0)} = \mathrm{Real}\displaystyle\sum_{m=1}^{\infty} w_{(0)m}\mathrm{e}^{2m\pi i\tau}$, where $w_{(0)m} = 3\dfrac{q_m^2 + 3q_m + 3}{(2\mu+1)q_m^2 + 9q_m + 9}\,u_m$ and $q_m = (1+i)\sqrt{m\pi\,\mathrm{Re}}$.

Let $N = 1$. Problem (2.8), (2.9) has the solution

$$v^{(1)}_{(1)r} = \frac{1}{r^2\sin\theta}\,\frac{\partial\bar\psi_{(1)}}{\partial\theta} + \tilde v_r, \quad v^{(1)}_{(1)\theta} = -\frac{1}{r\sin\theta}\,\frac{\partial\bar\psi_{(1)}}{\partial r} + \tilde v_\theta, \quad v^{(1)}_{(1)\varphi} = 0;$$

$$p^{(1)}_{(1)} = \Big[\frac{1}{\mathrm{Re}}\Big(\frac{\partial^3}{\partial r^3} - \frac{2}{r^2}\,\frac{\partial}{\partial r} + \frac{4}{r^3}\Big)\bar\psi_{(1)} - \frac{1}{r^2}\int_0^1 \frac{\partial^2\psi_{(0)}}{\partial r^2}\,\frac{da}{d\tau}\,d\tau\Big]\frac{\cos\theta}{\sin^2\theta} + \tilde p + c_{(1)};$$

$$\boldsymbol{w}^{(1)}_{(1)} = (\bar w_{(1)} + \tilde w_{(1)})\boldsymbol{k}. \tag{2.11}$$

Here $v^{(1)}_{(1)r}$, $v^{(1)}_{(1)\theta}$, and $v^{(1)}_{(1)\varphi}$ are, respectively, the r, θ, and φ components of the vector $\boldsymbol{v}^{(1)}_{(1)}$;

$$\bar\psi_{(1)} = \Big(-\frac{1}{2}\,\bar w_{(1)}r^2 + \frac{\alpha}{r} + \beta r + \Phi\Big)\sin^2\theta,$$

where

$$\alpha = -\frac{1}{4}\,\bar{w}_{(1)} + \frac{1}{2}\,(\lambda - 2\eta + \xi)\Big|_{r=1}, \quad \beta = \frac{3}{4}\,\bar{w}_{(1)} - \frac{1}{2}\,(\lambda + 3\xi)\Big|_{r=1}, \qquad \Phi = \frac{\eta}{r} + r^2\xi,$$

$$\lambda = \frac{1}{\sin^2\theta}\int\limits_0^1 \frac{\partial^2\psi_{(0)}}{\partial r^2}\,a\,d\tau, \quad \eta = \int\limits_\infty^r r\sigma\,dr, \qquad \xi = \int\limits_r^\infty \frac{\sigma}{r^2}\,dr, \qquad \sigma = \frac{1}{9}\,\mathrm{Re}\Big(r^3\int\limits_r^\infty \omega\,dr + \int\limits_\infty^r r^3\omega\,dr\Big),$$

$$\omega = \frac{1}{r^3\sin^2\theta}\int\limits_0^1 \Big(\frac{\partial^3\psi_{(0)}}{\partial r^3} - \frac{2}{r}\,\frac{\partial^2\psi_{(0)}}{\partial r^2} - \frac{2}{r^2}\,\frac{\partial\psi_{(0)}}{\partial r} + \frac{8}{r^3}\,\psi_{(0)}\Big)\frac{da}{d\tau}\,d\tau;$$

$$\tilde{v}_r = \mathrm{Real}\sum_{m=1}^{\infty} v_{rm}\mathrm{e}^{2m\pi i\tau}, \qquad \tilde{v}_\theta = \mathrm{Real}\sum_{m=1}^{\infty} v_{\theta m}\mathrm{e}^{2m\pi i\tau}, \qquad \tilde{p} = \mathrm{Real}\sum_{m=1}^{\infty} p_m\mathrm{e}^{2m\pi i\tau},$$

where v_{rm}, $v_{\theta m}$, and p_m are functions of r and θ; $c_{(1)}$ is a function of τ;

$$\bar{w}_{(1)} = \frac{1}{3}\,\mathrm{Real}\sum_{m=1}^{\infty} a_m^* u_m q_m^2$$

$$\times\Big[1 + \frac{1-\mu}{16}\Big(q_m^5 - q_m^4 + 14q_m^3 - 18q_m^2 + 48q_m + 48 - q_m^4(q_m^2+12)\mathrm{e}^{q_m}\int\limits_1^\infty \frac{\mathrm{e}^{-q_m r}}{r}\,dr\Big)\Big/((2\mu+1)q_m^2 + 9q_m + 9)\Big],$$

where a_m^* are constants, complex conjugate to a_m; $\tilde{w}_{(1)} = \mathrm{Real}\sum\limits_{m=1}^{\infty} w_{(1)m}\mathrm{e}^{2m\pi i\tau}$, where $w_{(1)m}$ are constants.

3. Using (2.10) and (2.11), we obtain

$$\boldsymbol{S} = \bar{W}t\boldsymbol{k} + \tilde{\boldsymbol{S}}, \tag{3.1}$$

where

$$\bar{W} = \frac{A_0}{T}\,\varepsilon æ \bar{w}_{(1)} \tag{3.2}$$

and $\tilde{\boldsymbol{S}} = \boldsymbol{S}_0 + \mathrm{Real}\sum\limits_{m=1}^{\infty} S_m\mathrm{e}^{2m\pi i\tau}\boldsymbol{k}$ ($\boldsymbol{S}_0$ and S_m are constants). Relation (3.1) gives an approximate dependence of $\boldsymbol{S}$ on t.

According to (3.1), the sphere moves along the Z axis, and its motion consists of oscillations and translational motion with constant velocity in the direction $\boldsymbol{k}$ (for $\bar{W} > 0$) or $-\boldsymbol{k}$ (for $\bar{W} < 0$). This means that the motion of the body in a specified direction is possible because of the pulsations of the rigid body and the oscillations of the fluid away from the body.

4. The conditions of the problem considered in the present paper are similar to those realized in the experiment of [1]. In both theory and experiment, similar oscillatory actions on a system consisting of a fluid and a rigid body result in motion of the body in the positive or negative direction of the axis along which the fluid oscillates away from the body. Because of this, the formulation of the problem and the results reported in the present paper can serve as a basis for a mathematical model of the phenomenon of predominantly unidirectional motion of a compressible rigid body in an oscillating fluid.

5. Let $a_1 \neq 0$, $u_1 \neq 0$, $a_n = 0$, and $u_n = 0$ $(n = 2, 3, \ldots)$. Using (3.2), we obtain

$$\frac{\bar{W}T}{\varepsilon æ A_0} \sim -\frac{2\pi}{9}\,\mathrm{Re}\,(\mu+2)\mathrm{Imag}\,(a_1^*u_1) + \frac{\pi^2}{162}\,\mathrm{Re}^2\,(\mu-1)(16\mu+35)\,\mathrm{Real}\,(a_1^*u_1) \quad \text{as} \quad \mathrm{Re} \to 0; \tag{5.1}$$

$$\frac{\bar{W}T}{\varepsilon æ A_0} \sim -\frac{2\pi}{3}\,\mathrm{Re}\,\mathrm{Imag}\,(a_1^*u_1) + 5\,\frac{\mu-1}{2\mu+1}\,\mathrm{Real}\,(a_1^*u_1) \quad \text{as} \quad \mathrm{Re} \to \infty. \tag{5.2}$$

In particular, from (5.1) and (5.2), it follows that if $\mathrm{Imag}\,(a_1^*u_1) = 0$, the directions of motion of the sphere are opposite for $\mu < 1$ and $\mu > 1$ and there is no motion for $\mu = 1$.

Thus, for the same pulsations of the body and oscillations of the fluid away from the body, the behavior of the body can be qualitatively different, depending on the ratio of its average density to the density of the fluid.

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